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DIFFERENTIAL FORM OF THE UNIVERSAL EQUATION OF THE LAMINAR BOUNDARY LAYER

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We propose a new approach to composing a universal equation of the laminar boundary layer in generalized similarity variables.

§1. The wide use of electronic digital calculators has greatly reduced interest in approximate methods of ; of computation. However, the problem of establishing general rules to describe the effect of factors external to the boundary layer (such as the velocity distribution at the outer boundary, blowing or suction velocities, body surface temperatures, external magnetic field stresses, etc.) on terminal characteristics (friction stress, heat-transfer coefficient, flow separation location, etc.) continues to be one of practical and fundamental significance. These rules express general tendencies of various processes such as flow drag, heat transfer, and related motions in boundary layers.

The "generalized similarity method," proposed in [1] by Loitsyanskii, makes it possible to examine broad classes of problems of boundary-layer theory by transforming the boundary-layer equation to a "universal" generalized-similarity form requiring only a single numerical integration. The resulting tables of solutions, prepared once and for all, express general rules and relationships among the basic characteristics of the boundary layer.

§2. In its initial form the generalized similarity method was first published in [1]. Its distinguishing feature was that its basic universal equation was of integrodifferential form, in which differential and integral functionals of the unknown solution were present. In the rather simple cases treated in that paper only minor complications were encountered in numerically integrating the fundamental equation. Results of the integration and a corresponding bibliography can be found in [2].

The attempt to apply the method to more involved cases (nonstationary boundary layer, jets and wakes in arbitrary pressure fields, etc.) showed that by reducing the universal equations to purely differential form one could, in spite of the introduction thereby of an increase in the number of independent variables, significantly simplify the form of the universal equation and aid in effecting the first stage of the method, namely, that of deriving general rules. In the present paper we develop the basic notion involved in the transition of the universal equation from an integrodifferential form to one of purely differential form, and we apply it to a physically realistic and sufficiently general example of a two-dimensional high-speed boundary layer in a homogeneous incompressible fluid; generalization of the notion to more involved motions is then perfectly straightforward.

§3. As a direct substitution of an affinely similar form of the stream function $\psi = U\delta\psi_1(y/\delta)$ into the general Prandtl equation reveals, there appears in the equation a particular pair of conjugate parameters: $f_1 = U'z$, $\overline{f_1} = Uz'$ which are explicit functions of x [U(x) is the speed at the outer edge of the boundary layer; $z = \delta^2/\nu$; δ is a conditional "thickness" of the boundary layer; and the primes indicate differentiation with respect to x], thereby violating the universality of the new form of the equation. Introduction of these parameters into a number of the independent [by virtue of the arbitrariness of U(x)] variables, i.e., a transition

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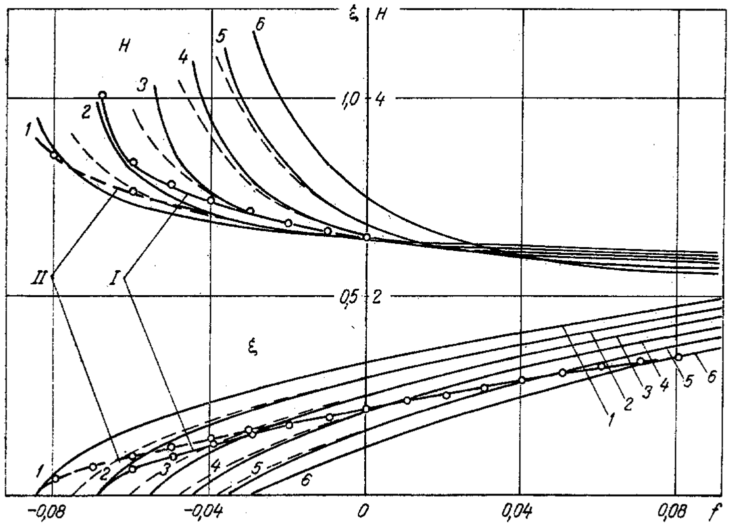

Fig. 1. Dependence of ξ and H on f (curves 1, 2, 3, 4, 5, and 6 are for values of $\bar{f} = 1$, 0.8, 0.6, 0.4, 0.2, and 0.0, respectively); the solid curves indicate a locally similar approximation, while the dashed curves indicate a locally two-parameter approximation.

to a generalized similarity form of the stream function $\psi = U\delta\psi_2(y/\delta, f_1, \bar{f}_1)$, and then again substituting this expression into the Prandtl equation, leads to the appearance of a "residual" consisting of a new pair of conjugate parameters: $f_2 = UU''z^2$, $\bar{f}_2 = U^2zz''$, and so forth. But the transition to two infinite sequences of variables (the question as to the convergence of the method remains open), namely,

$$f_k = U^{k-1}\frac{d^kU}{dx^k}z^k, \quad \bar{f}_k = U^k z^{k-1}\frac{d^kz}{dx^k} \quad (k = 1, 2, \ldots) \tag{1}$$

and to an expression for the stream function in the form

$$\psi = U\delta\Phi[\eta;\ (f_k),\ (\bar{f}_k)], \quad \eta = y/\delta \tag{2}$$

[we employ here the notation (f_k), $(\bar{f}_k)$ for the sequences $(f_1, f_2, \ldots)$ and $(\bar{f}_1, \bar{f}_2, \ldots)$ of the generalized similarity parameters], enables us to transform the Prandtl equation, expressed in terms of the stream function ψ, into the following purely differential equation of the boundary layer in the generalized similarity variables:

$$\frac{\partial^3\Phi}{\partial\eta^3} + \left(f_1 + \frac{1}{2}\bar{f}_1\right)\Phi\frac{\partial^2\Phi}{\partial\eta^2} + f_1\left[1 - \left(\frac{\partial\Phi}{\partial\eta}\right)^2\right] = \sum_{(k)}\left[\theta_k\frac{D(\Phi,\ \partial\Phi/\partial\eta)}{D(\eta,\ f_k)} + \bar{\theta}_k\frac{D(\Phi,\ \partial\Phi/\partial\eta)}{D(\eta,\ \bar{f}_k)}\right]. \tag{3}$$

In deriving this equation we have used the recursion relations

$$\begin{aligned} \frac{U}{U'}f_1 f_k' &= \theta_k = [(k-1)f_1 + k\bar{f}_1]f_k + f_{k+1}, \\ \frac{U}{U'}\bar{f}_1\bar{f}_k' &= \bar{\theta}_k = [(k-1)\bar{f}_1 + kf_1]\bar{f}_k + \bar{f}_{k+1}, \end{aligned} \tag{4}$$

which follow directly from the definition of f_k and $\bar{f}_k$. To the equation (3) we adjoin the boundary conditions (the overdot indicates differentiation with respect to η)

$$\Phi = \dot{\Phi} = 0 \ \text{ for } \ \eta = 0, \ \dot{\Phi} \to 1 \ \text{ for } \ \eta \to \infty, \ \Phi = \Phi_0(\eta) \ \text{ for } \ (f_k) = (f_{k0}) = \text{const}, \ (\bar{f}_k) = (\bar{f}_{k0}) = \text{const}. \tag{5}$$

The first two boundary conditions refer to the case of the boundary layer on an impermeable solid wall; the last condition displays the fact that $\Phi_0(\eta)$ represents the solution of the self-similar Falkner–Skan problem

$$\begin{gathered} \dddot{\Phi}_0 + \left(f_{10} + \frac{1}{2}\bar{f}_{10}\right)\Phi_0\ddot{\Phi}_0 + f_{10}(1 - \dot{\Phi}_0^2) = 0, \\ \Phi_0 = \dot{\Phi}_0 = 0, \ \eta = 0, \ \dot{\Phi}_0 \to 1, \ \eta \to \infty, \end{gathered} \tag{6}$$

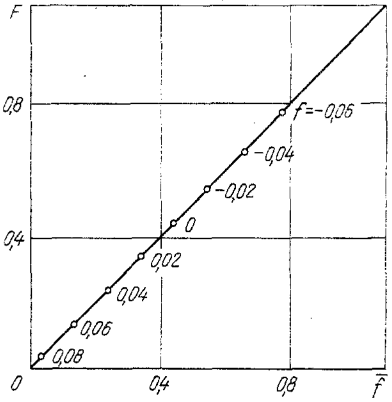

Fig. 2. Determination of the dependence $F(f, \bar{f})$; ξ, H, f, $\bar{f}$, and F are dimensionless quantities.

corresponding to the external velocity distribution $U(x) = cx^m$. The constants f_{k0} and $\bar{f}_{k0}$ are given by

$$f_{k0} = \frac{2^k}{(m+1)^k} m(m-1)\ldots(m-k+1)[B(\beta)]^{2k},$$
$$\bar{f}_{k0} = \frac{(-1)^k 2^k}{(m+1)^k}(m-1)m\ldots(m+k-2)[B(\beta)]^{2k}, \tag{7}$$
$$\beta = \frac{2m}{m+1};$$

a table of $B(\beta)$ is given in [2].

The last row of the boundary conditions (5) represents the principal limitation to the generality of the statement of boundary-layer theory problems with an arbitrarily assigned velocity profile in the "initial" section of the layer (the problem of "continuation"). This restriction is deeply connected with the generalized similarity concept itself; to remove it would require further additional generalizations.

Equation (3), with the boundary conditions (5), constitutes a nonlinear third-order partial-differential equation with the independent variables η, (f_k), $(\bar{f}_k)$; it possesses the property of universality in the sense that it, and the boundary conditions corresponding to it, have one and the same form for arbitrary analytically specified velocity distributions U(x) at the outer edge of the boundary layer.

Equation (3), subject to the boundary conditions (5), can be integrated once and for all on an electronic digital computer, where, with contemporary machine potentialities, we can only speak of the "section" of the equation corresponding to the value k = 1, since, even in this case, the equation contains the three variables: η, f_1, $\bar{f}_1$.

Thus, Eq. (3), with the boundary conditions (5), can be integrated numerically only in the simplest approximation ($k = 1; f_1 = f, \bar{f}_1 = \bar{f}$):

$$\frac{\partial^3\Phi}{\partial\eta^3} + \left(f + \frac{1}{2}\bar{f}\right)\Phi\frac{\partial^2\Phi}{\partial\eta^2} + f\left[1 - \left(\frac{\partial\Phi}{\partial\eta}\right)^2\right]$$
$$= f\bar{f}\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial^2\Phi}{\partial\eta\partial f} - \frac{\partial\Phi}{\partial f}\frac{\partial^2\Phi}{\partial\eta^2} + \frac{\partial\Phi}{\partial\eta}\frac{\partial^2\Phi}{\partial\eta\partial\bar{f}} - \frac{\partial\Phi}{\partial\bar{f}}\frac{\partial^2\Phi}{\partial\eta^2}\right),$$
$$\Phi = \dot{\Phi} = 0 \text{ for } \eta = 0, \; \dot{\Phi} \to 1 \text{ for } \eta \to \infty, \tag{8}$$
$$\Phi = \Phi_0(\eta) \text{ for } f = f_0 = \text{const}, \; \bar{f} = \bar{f}_0 = \text{const}.$$

We speak of this as a full two-parameter approximation. If, in addition, we discard on the right side of Eq. (8) derivatives with respect to the variable $\bar{f}_1$, retaining it as a parameter (we speak of such an approximation as being local in the variable $\bar{f}_1$), we then obtain the following equation in a locally two-parameter approximation:

$$\frac{\partial^3\Phi}{\partial\eta^3} + \left(f + \frac{1}{2}\bar{f}\right)\Phi\frac{\partial^2\Phi}{\partial\eta^2} + f\left[1 - \left(\frac{\partial\Phi}{\partial\eta}\right)^2\right] = f\bar{f}\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial^2\Phi}{\partial\eta\partial f} - \frac{\partial\Phi}{\partial f}\frac{\partial^2\Phi}{\partial\eta^2}\right), \tag{9}$$

$$\Phi = \dot{\Phi} = 0 \quad \text{for} \quad \eta = 0, \quad \dot{\Phi} \to 1 \quad \text{for} \quad \eta \to \infty,$$

$$\Phi = \Phi_0(\eta) \quad \text{for } f = f_0 = \text{const}, \quad \bar{f} = \bar{f}_0 = \text{const}.$$

In the event that on the right side of Eq. (8) we discard derivatives with respect to both of the parameters f and $\bar{f}$, we shall have a locally similar approximation. In this approximation the solution of the equation is readily obtainable with a simple application of a solution of the Falkner–Skan equation.

§4. Then termwise integration of both sides of Eq. (8) with respect to η from $\eta = 0$ to $\eta = \infty$ yields, if we take as the "thickness" $\delta(x)$, the "momentum loss" δ^{**}, where

$$\delta^{**}(x) = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy, \tag{10}$$

the following Kármán integral relation expressed in the variables f and $\bar{f}$:

$$\bar{f} = 2[\xi - (2 + H) f] = F(f, \bar{f}), \tag{11}$$

wherein

$$\xi = \left[\frac{\partial (u/U)}{\partial (y/\delta^{**})}\right]_{y=0} = \ddot{\Phi}(0;\, f, \bar{f}) \quad H = \int_0^\infty (1 - \dot{\Phi})\, d\eta = \frac{\delta^*}{\delta^{**}}. \tag{12}$$

Outwardly, the integral relation (11) has the usual form, but now the variables ξ and H in it are functions of the variables f and $\bar{f}$. Replacing f and $\bar{f}$ by their expressions $f = U'z$, $\bar{f} = Uz'$, we obtain an ordinary nonlinear differential equation in

$$V(x) \frac{dz^{**}}{dx} = F(U'z^{**}, Uz^{**\prime}). \tag{13}$$

The initial condition $z^{**} = z_0^{**}$ at $x = x_0$ expresses integrally the distribution of velocities at the initial section of the boundary layer.

Equations (12), for the case in question of a high-speed boundary layer, constitute the desired general rules for the relationship involving the "reduced" exit characteristics: ξ (the reduced frictional stress on the surface), H (the degree of "completeness" of the velocity profiles at sections of the boundary layer), and the parameters f and $\bar{f}$, which describe features of the geometric shape of the curves for the distribution of the "external" velocity and for the "thickness" of the boundary layer. Solution of the ordinary differential equation (13) in a specific case where U(x) is specified [compatible with use of tables of $\dot{\Phi}(\eta; f, \bar{f})$, $\ddot{\Phi}(0; f, \bar{f})$] makes it possible to obtain an approximate solution of a particular given problem.

§5. It is easy to establish a relationship between the new and the old methods of solving the boundary-layer equation. Reduction of the system of variables $(f_k, \bar{f}_k)$ to the "truncated" system $(f_k, \bar{f}_1)$, and then "localization" with respect to $\bar{f}_1$, reduces Eq. (3) to the form

$$\frac{\partial^3 \Phi}{\partial \eta^3} + [\xi - (1 + H) f_1]\, \Phi \frac{\partial^2 \Phi}{\partial \eta^2} - f_1 \left[1 - \left(\frac{\partial \Phi}{\partial \eta}\right)^2\right] = \sum_{(k)} \theta_k \frac{D(\Phi, \partial\Phi/\partial\eta)}{D(\eta, f_k)}, \tag{14}$$

$$\Phi = \dot{\Phi} = 0 \quad \text{for} \quad \eta = 0, \quad \dot{\Phi} \to 1 \quad \text{for} \quad \eta \to \infty,$$

$$\Phi = \Phi_0(\eta) \quad \text{for} \quad f_l = f_{l0} = \text{const}, \quad \bar{f}_l = \bar{f}_{l0} = \text{const}.$$

Equation (14) constitutes an integrodifferential (see [1]) form of the universal equation in which $\xi[(f_k)]$ is a differential functional and $H[f_k]$ an integral functional of the desired solution.

Thus, the old method is a particular case of the new method and corresponds to a "truncation" over all the "conjugate" variables $(\bar{f}_k)$, except for the first variable $\bar{f}_1$, and then a localization with respect to this variable.

The new method is more general but, owing to limitations of electronic digital computers, is apparently applicable only in the particular case corresponding to the value $k = 1$ [Eq. (8)]. This type of equation can be used even in the more general case when solving the universal equation for motions close to being self-similar, when there is an effective method for expanding in series (see [3]) about the values $(f_{k0}, \bar{f}_{k0})$ corresponding to the self-similar Falkner—Skan equation.

§6. We obtained a numerical solution of Eq. (3) in the locally similar and two-parameter cases, the approximations being local with respect to the parameter $\bar{f}_1$. The integration was carried out on the M-220 electronic digital computer by the method of finite differences.

Figure 1 shows the solution in the form of the dependence of ξ and H on f for various fixed values of the parameter $\bar{f}$ in both approximations; in Fig. 2 the solution is shown for the combination $F = 2[\xi - (2+H)f]$ in the locally similar approximation (the corresponding graph in the locally two-parameter approximation is similar).

The locally similar approximation of the "universal" equation (3) corresponds to the locally one-parameter approximation in the old method (Kochin—Loitsyanskii; curves I); the two-parameter approximation, local with respect to the parameter $\bar{f}_1$, corresponds to the complete one-parameter approximation (Howarth; curves II). The solution obtained by the new method can be compared with the solution obtained in [1]. To do this, we need to solve, in accordance with the new method, the transcendental equation

$$\bar{f} = F(f, \bar{f}); \tag{15}$$

the latter makes it possible to relate the variables f and $\bar{f}$ whenever the solution of the universal equation, in this or another approximation, is known. This is best done graphically by taking $\bar{f}$ along the axis of abscissas and F along the axis of ordinates and then taking as the solution the points of this plane lying on the angle bisector (see Fig. 2). As a result, for each approximation we will have a dependence of $\bar{f}$ on f.

Upon comparing the distributions $\xi[f, \bar{f}(f)] = \xi(f)$, $H[f, \bar{f}(f)] = H(f)$, $F[f, \bar{f}(f)] = F(f)$ with the locally similar solution and the complete one-parameter solution by the old method, we see that they are equivalent.

We remark that the existing two-parameter solution [2] of the old method corresponds to the new four-parameter solution, "truncated" in $\bar{f}_2$ and "local" with respect to $\bar{f}_1$.

Our main results are the following: the derivation of a purely differential form of the universal equation and the establishing of a relationship between the old and the new methods.

Along with this, we point out the completely rational way of introducing the parameters, as described in Sec. 3.

Upon considering the various approximate forms of the universal equation, obtained through "truncation" and "localization," we see that there are two ways of solving the problem: reducing the equation to an integro-differential form (the "old" method) or to a purely differential form (the "new" method). In physically involved problems (nonstationary boundary layer, a case of several arbitrary velocity scales, and so forth) the new method is preferable.

NOTATION

x, y, longitudinal and transverse coordinates in a boundary layer; η, dimensionless-transverse coordinate; U, velocity on the external boundary of a boundary layer; ψ, stream-function; Φ, dimensionless stream-function; u, velocity projection into the x axis in a boundary layer; ν, kinematic viscosity coefficient; $\delta(x)$, some conventional thickness of a boundary layer; δ^*, displacement thickness; δ^{**}, momentum thickness; H, characteristic function; ξ, reduced friction coefficient; $f_k, \bar{f}_k$, dimensionless parameters.

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